Tropical Methods in $\mathbb{A}^1$-enumerative Geometry

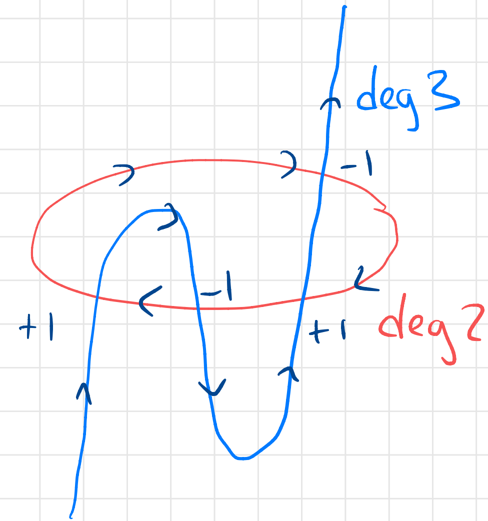
jt with Andrés Jaramillo Puentes

Today: Bézout's theorem for curves

Bézout's theorem for curves $/\mathbb{C}$

$C_1 = V(F_1) \subseteq \mathbb{P}^2 \qquad d_1 = \deg F_1$

$C_2 = V(F_2) \subseteq \mathbb{P}^2 \qquad d_2 = \deg F_2$



Then
$$\sum_{p \in C_1 \cap C_2} \mathrm{mult}_p(C_1, C_2) = d_1 d_2$$

Over $\mathbb{R}$:

If $d_1 + d_2 \equiv 1 \mod 2$

then
$$\sum_{p \in C_1 \cap C_2} \mathrm{sign}_p(C_1, C_2) = 0$$

where $\mathrm{sign}_p(C_1, C_2) = \mathrm{sign}\det \mathrm{Jac}(F_1, F_2)(p)$

Question: What about other fields k?

Grothendieck-Witt ring: $\quad k$ field $\qquad \langle a \rangle = a x^2$

$GW(k)$ is generated by $\langle a \rangle$ where $a \in k^\times / (k^\times)^2$

relations:

1) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle \qquad ab(a+b) \neq 0$

2) $\langle a \rangle \langle b \rangle = \langle ab \rangle \qquad ab \neq 0$

3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle =: h \qquad a \neq 0$

hyperbolic form

$$\begin{array}{c} V \xrightarrow{q} k(p) \\ \big\downarrow \mathrm{Tr}_{k(p)/k} \\ k \end{array}$$

Examples:

- $GW(\mathbb{C}) = \mathbb{Z}$
- $GW(\mathbb{R}) = \{ (r,s) \in \mathbb{Z} \times \mathbb{Z} : r+s \equiv 0 \mod 2 \}$

Thm (McKean): $C_i = V(F_i) \subseteq \mathbb{P}^2_k$, $d_i = \deg F_i$, $i = 1, 2$

If $d_1 + d_2 \equiv 1 \mod 2$, then

$\mathbb{A}^1$-homotopy theory

$$\sum_{p \in C_1 \cap C_2} \mathrm{Tr}_{k(p)/k} \langle \det \mathrm{Jac}(F_1, F_2)(p) \rangle = \frac{d_1 \cdot d_2}{2} \cdot h \in GW(k)$$

Motivation: Tropical geometry

Start with polynomials

$F_1(x,y) = \sum a_{i_1 i_2} x^{i_1} y^{i_2}$

$F_2(x,y) = \sum b_{j_1 j_2} x^{j_1} y^{j_2}$

Toric deformations:

$F_1^t(x,y) = \sum a_{i_1 i_2} x^{i_1} y^{i_2} t^{\varphi(i_1,i_2)}$

$F_2^t(x,y) = \sum b_{j_1 j_2} x^{j_1} y^{j_2} t^{\psi(j_1,j_2)}$

$\psi, \varphi : (\mathbb{Z}_{\geq 0})^2 \to \mathbb{Q}$

$\in k\{\{t\}\}[x,y]$

Puiseux series

$k\{\{t\}\} = \left\{ \sum_{i=i_0}^{\infty} c_i t^{i/n} : n \in \mathbb{N}, c_i \in k \right\}$

valuation:

$\mathrm{val}: k\{\{t\}\} \to \mathbb{R} \cup \{\infty\}$

$c_{i_0} t^{i_0/n} + \mathrm{hot} \mapsto i_0/n$

$-\mathrm{val}(x+y) \leq \max(-\mathrm{val}\, x, -\mathrm{val}\, y)$

$-\mathrm{val}(x \cdot y) = -\mathrm{val}\, x + -\mathrm{val}\, y$

$\rightsquigarrow$ tropical semi-field $(\mathbb{T}, "+", "\cdot")$

$\mathbb{T} = \mathbb{R} \cup \{-\infty\}$

$"x+y" = \max(x,y)$

$"x \cdot y" = x + y$

Tropical curves

tropical polynomial

$$f(x,y) = \text{"}\sum c_{i,j}\, x^i y^j\text{"} = \max\{c_{ij} + ix + jy\} \in \mathbb{T}[x,y]$$

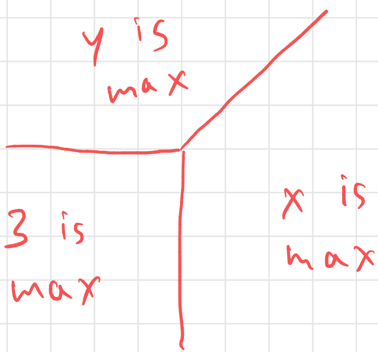
tropical curve

$$V(f) = V_{trop}(f) = \{(x,y) \in \mathbb{R}^2 : \max \text{ is attained twice}\}$$

Example: tropical line

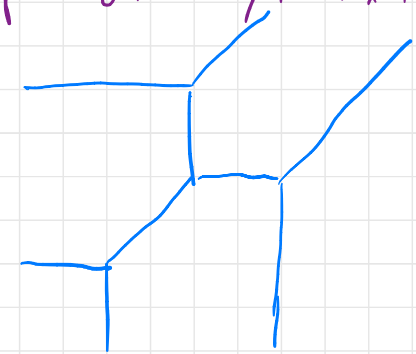
$$f = \text{"}x + y + 3\text{"}$$

$$= \max(x, y, 3)$$



Example: tropical conic

$$f = \text{"}0 + x + y + (-4)x^2 + (-2)xy + (-4)y^2\text{"}$$

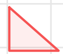


$f = \text{"}\sum c_{i,j} x^i y^j\text{"} \in \mathbb{T}[x,y]$

Newton polygon $\Delta(f) := \mathrm{Conv}\{(i,j) \in (\mathbb{Z}_{\geq 0})^2 \mid c_{i,j} \neq -\infty\} \subseteq \mathbb{R}^2$

Ex: line : $f = \text{"}x + y + 3\text{"}$

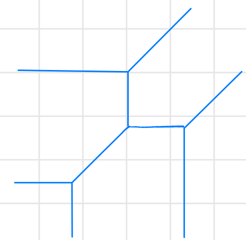
$\Delta(f) = \mathrm{Conv}((1,0),(0,1),(0,0))$

$=$  $= \Delta_1$

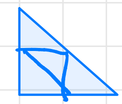
$\Delta_d := \mathrm{Conv}\{(d,0),(0,d),(0,0)\}$

Ex: Conic

$f = \text{"}0 + x + y + (-4)x^2 + (-2)xy + (-4)y^2\text{"}$



$\Delta(f) = \Delta_2$



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dual subdivision $SD(f)$:

vertices of $V(f)$ $\longleftrightarrow$ max cells of $SD(f)$

edges of $V(f)$ $\longleftrightarrow$ edges of $SD(f)$

connected components of $\mathbb{R}^2 \setminus V(f)$ $\longleftrightarrow$ vertices of $SD(f)$

s.t.

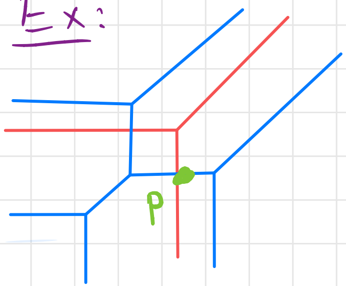
- all inclusions are inverted
- dual edges are orthogonal to each other

Bézout for tropical curves

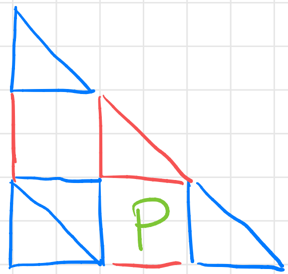
Let $f_1, f_2 \in \mathbb{T}[x,y]$. $C_1 := V(f_1)$, $C_2 := V(f_2)$

Let $p \in C_1 \cap C_2$. Then p corresponds to a Parallelograms in the dual subdivision of $C_1 \cup C_2$.

Ex:



dual subdivision



Thm (Bézout for tropical curves)

Assume $\Delta(f_1) = \Delta_{d_1}$, $\Delta(f_2) = \Delta_{d_2}$

$$\sum_{p \in C_1 \cap C_2} \mathrm{mult}_p(C_1, C_2) = d_1 \cdot d_2$$

Def:

$\mathrm{mult}_p(C_1, C_2) :=$ Area of parallelogram in dual subdiv of $C_1 \cup C_2$

$=$ # solutions to $F_1^t = F_2^t = 0$ st $-\mathrm{val} = p$

Proof (Sturmfels): $\sum_{p \in C_1 \cup C_2}$ Area parallelograms

$= \text{Area } \Delta_{d_1+d_2} - \text{Area } \Delta_{d_1} - \text{Area } \Delta_{d_2}$

$= \frac{(d_1+d_2)^2}{2} - \frac{d_1^2}{2} - \frac{d_2^2}{2} = d_1 \cdot d_2$ $\square$

Question: Can we do this in $GW(k)$?

$GW(k)$

$\|$

Idea: $\widetilde{\mathrm{mult}}_p(C_1, C_2) := \mathrm{Tr}_{E/k\{\{t\}\}} \langle \det \mathrm{Jac}(F_1^t, F_2^t)(z) \rangle \in GW(k\{\{t\}\})$

$F_1^t = \sum a_{i_1 i_2} x^{i_1} y^{i_2} t^{\varphi(i_1, i_2)}$

$F_2^t = \sum b_{j_1 j_2} x^{j_1} y^{j_2} t^{\psi(j_1, j_2)}$

coord ring all such z

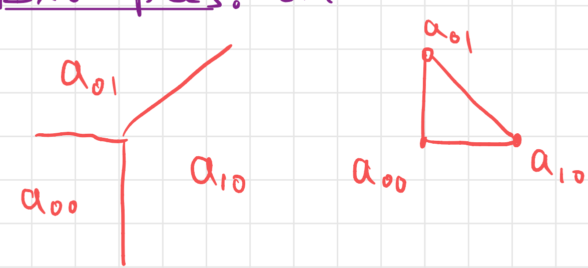
solution to $F_1^t = F_2^t = 0$ st $-\mathrm{val}(z) = p$

Problem: This depends on $a_{i_1 i_2}$ & $b_{j_1 j_2}$

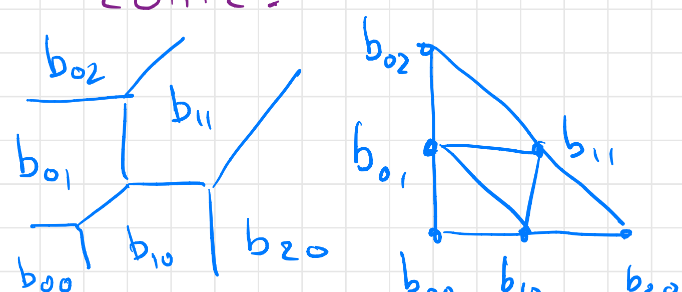
Def: enriched tropical curves$/k$ (Viro's patchworking)

= tropical curves with elmts in $k^\times/(k^\times)^2$ assigned to each comp

Examples: line:



conic:



Combinatorial formula for $\widetilde{\mathrm{mult}}_p(C_1, C_2)$

Def: We say that $v \in \mathbb{Z}^2$ is **odd** if $v = (1,1)$ in $(\mathbb{Z}/2)^2$

Thm (Jaramillo Puentes - P.)

P = parallelogram in dual subdiv of $C_1 \cup C_2$ dual to $p \in C_1 \cap C_2$

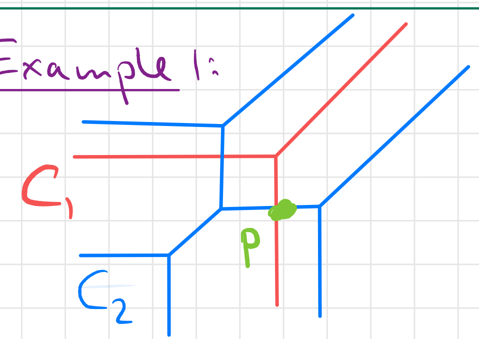
$$\widetilde{\mathrm{mult}}_p(C_1, C_2) = \sum_{\substack{\text{odd vertices}\\ \text{of } P}} \langle \varepsilon(v)\, a_v \rangle + \frac{\text{Area } P - \#\text{ odd vertices of } P}{2} \cdot h \in GW(k)$$

coeff of v

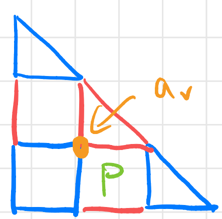
$$\varepsilon(v) = \begin{cases} +1 & C_1 \text{ first} \\ -1 & C_2 \text{ first} \end{cases}$$

walk around v in P anticlockwise

Example 1:



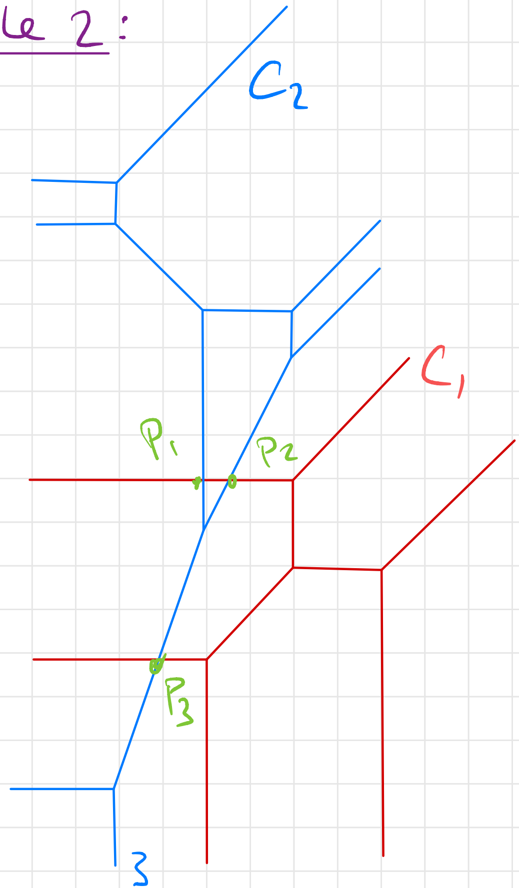
dual subdiv



$$\widetilde{\mathrm{mult}}_p(C_1, C_2) = \langle -a_v \rangle$$

$$\langle a_v \rangle + \langle -a_v \rangle = h = \langle 1 \rangle + \langle -1 \rangle$$

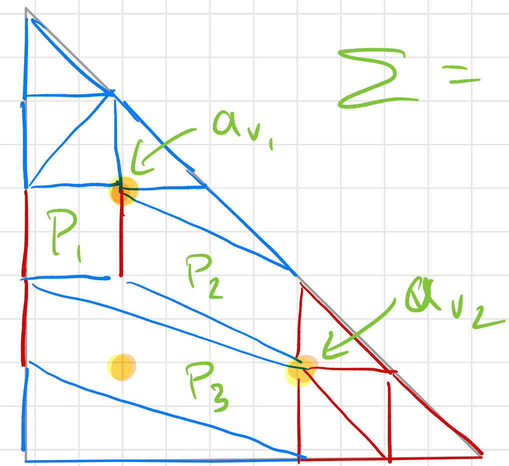
Example 2:



$$\widetilde{\mathrm{mult}}_{p_1}(C_1, C_2) = \langle -a_{v_1} \rangle$$

$$\widetilde{\mathrm{mult}}_{p_2}(C_1, C_2) = \langle a_{v_1} \rangle + \langle a_{v_2} \rangle$$

$$\widetilde{\mathrm{mult}}_{p_3}(C_1, C_2) = \langle -a_{v_2} \rangle + h$$



$$\Sigma = 3h$$

Cor: (Bézout of enriched tropical curves)

$d_1 + d_2 \equiv 1 \mod 2$

Then $$\sum_{p \in C_1 \cap C_2} \widetilde{\text{mult}}_p (C_1, C_2) = \frac{d_1 \cdot d_2}{2} \cdot h \in GW(k)$$

Proof:

- If $d_1 + d_2 \equiv 1 \mod 2$ then there is <u>no</u> odd vertex on the boundary of $\Delta_{d_1+d_2}$
- Let v be a vertex in the interior of $\Delta_{d_1+d_2}$

1) # Parallelograms corres to an intersection with v as a vertex is even

2) $\#\{P : v \text{ is a vertex}, \ \varepsilon(v) = +1\}$

$= \#\{ \quad \text{——} \quad , \quad \text{——} \quad -1\}$

$\langle a_v \rangle + \langle -a_v \rangle = h$ + classical tropical Bézout $\Rightarrow$ Proof $\square$

Generalizations & more results

- can define enriched tropical hypersurfaces in any dimension
  $\rightsquigarrow$ enriched tropical Bézout (not just for curves)
  $\Rightarrow$ new proof of Bézout's theorem enriched in $GW(k)$
- Can count intersections in any toric variety
  $\rightsquigarrow$ enriched Bernstein - Kushnirenko
- Can also say something about the possible counts in non-relatively orientable case.

THANK YOU !